Explosive Instability in Superposed Fluids in Hydromagnetics

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We study the resonance interaction between two marginally unstable modes of small but finite amplitude waves in the Kelvin-Helmholtz flow model using the method of multiple scales. We find that, as magnetic field increases, the amplitudes of the disturbance diverge at a very high rate while for a fluid of higher density ratio ϱ, the amplitudes diverge very slowly.

1. Introduction

Explosive instability of Kelvin-Helmholtz flow has been investigated by several authors in recent years (see: Murakami [1]; Adam and Craik [2]). It was pointed out by Weissman [3] that the time-dependent amplitude equation for waves anywhere on the surface fails if $k^2 = g/2\sigma = k_{res}^2$, where k is the wavenumber, g is the gravitational force, σ the surface tension and k_{res} the resonant wavenumber. In this case the non-linear coefficients of the amplitude equation have a singularity and hence it diverges. This is because not only the fundamental mode satisfies the dispersion relation, but the second harmonic mode also does, in other words, second harmonic resonance takes place. In that case we must take $A_2(t_1)\exp[2i(kx_0 - \omega t_0)]$ in addition to the amplitude $A_1(t_1)\exp[i(kx_0 - \omega t_0)]$ in order to describe the nonlinear interaction between the fundamental mode and the second harmonic resonance mode. In order to study such problems, Weissman [3] suggested a special scaling procedure, which we discuss in the next section. This way we obtain the appropriate form of the equation.

Nayfeh and Saric [4] considered the second harmonic resonance of two superposed fluids moving with uniform velocities. They obtained the equation of first order in time, similar to the one obtained by McGoldrick [5]. This equation holds in the stable region, but as the neutral curve is approached, the nonlinear coefficients become singular. This is due to the vanishing of the derivative of the characteristic function (dispersion relation) with respect to the frequency ω on the neutral surface. Murakami [1] followed Weissman's [3] scaling procedure and calculated the nonlinear coefficients. Coupled amplitude equations were obtained, which are both time and space dependent. However, if one ignores the spatial dependence of the amplitudes, the problem can be examined numerically and one finds that explosive instability takes place, in which the amplitudes of both modes grow rapidly. Most of the research work done in the nonlinear theory pertains to irrotational motions. However there has been some progress in the general case, allowing for rotational flow (see: Lardner and Trehan [6]; Chhabra and Trehan [7]). In this paper we consider the effect of a tangential magnetic field on the second harmonic resonance on the marginally neutral curve at the interface between two incompressible, inviscid fluids.

2. Formulation of the Problem

We consider the two dimensional flow of two semi-infinite homogeneous superposed fluids separated by the interface $z = 0$. The fluids are assumed to be incompressible and inviscid. The lower fluid of density $\varrho^{(1)}$ is assumed to be at rest, while the upper fluid of density $\varrho^{(2)}$ is moving along the x-axis with uniform speed U. Uniform magnetic fields H_i $(i = 1, 2)$ are imposed on the lower and the upper fluid, respectively. The system is under the influence of gravity $\boldsymbol{g}(0, 0, -1)$. The equations governing the system are

$$\frac{\partial \boldsymbol{u}^{(i)}}{\partial t} + (\boldsymbol{u}^{(i)} \cdot \nabla)\boldsymbol{u}^{(i)} = -\nabla \Pi^{(i)} + (\boldsymbol{h}^{(i)} \cdot \nabla)\boldsymbol{h}^{(i)} + \boldsymbol{g}, \quad (1)$$

$$\frac{\partial \boldsymbol{h}^{(i)}}{\partial t} = (\boldsymbol{h}^{(i)} \cdot \nabla)\boldsymbol{u}^{(i)} - (\boldsymbol{u}^{(i)} \cdot \nabla)\boldsymbol{h}^{(i)}, \quad (2)$$

$$\nabla \cdot \boldsymbol{u}^{(i)} = 0, \quad \nabla \cdot \boldsymbol{h}^{(i)} = 0, \quad (3)$$

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where

$$\Pi^{(i)} = \frac{p^{(i)}}{\varrho^{(i)}} + \frac{1}{2}(\boldsymbol{h}^{(i)})^2.$$

The corresponding boundary conditions at the free surface $z = \eta(x, t)$ are

$$\frac{\partial \eta}{\partial t} + u^{(i)} \frac{\partial \eta}{\partial x} - w^{(1)} = 0, \tag{4}$$

$$\frac{\partial \eta}{\partial t} + (U + u^{(2)}) \frac{\partial \eta}{\partial x} - w^{(2)} = 0, \tag{5}$$

$$\boldsymbol{n} \cdot \boldsymbol{h}^{(1)} = \boldsymbol{n} \cdot \boldsymbol{h}^{(2)}, \tag{6}$$

$$(\varrho^{(1)} \Pi^{(1)} - \varrho^{(2)} \Pi^{(2)}) - g\,\eta\,(\varrho^{(1)} - \varrho^{(2)}) + \sigma \frac{\partial^2 \eta}{\partial x^2}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-3/2} = 0, \tag{7}$$

and the vanishing of the normal component of the velocity and the magnetic field at $z = -\infty$ and $z = \infty$. Here σ, $\boldsymbol{n}$, $\boldsymbol{p}^{(i)}$, $\boldsymbol{u}^{(i)}$, $\boldsymbol{h}^{(i)}$ denote the coefficient of surface tension, the unit outward normal, the pressure, the perturbed velocity and the magnetic field, respectively.

We employ the method of multiple scales as formulated by Weissman [3] and Murakami [1] to describe the resonance interaction between two marginally stable modes of small but finite amplitude waves. We introduce a small parameter $\varepsilon = (u/u_m - 1)^{1/2}$, which represents the departure of the system from the bifurcation value u_m. This bifurcation value, obtained in the next section, is such that the difference between the original velocity u and the final velocity u_m is of the order of ε^2, $\varepsilon \ll 1$. In other words, we discuss here the stability of the system in the neighbourhood of the neutral curve by setting the bifurcation parameter u as $u_m(1 \pm \varepsilon^2)$.

In order to have significant interaction at such a resonance, and also in order to separate the rapid change of the phase and the slow change of the amplitude of the carrier wave, we employ the method of multiple scales by introducing $x_n = \varepsilon^n x$, $t_n = \varepsilon^n t$, $(n = 0, 1, 2)$. The differential operators in various equations may be expanded as follows:

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \alpha_0} + \varepsilon \frac{\partial}{\partial \alpha_1} + \varepsilon^2 \frac{\partial}{\partial \alpha_2} + \ldots, \tag{8}$$

where α is any of the variables x or t. Since we wish to describe the nonlinear interactions in a slightly unstable region, we expand the various physical quantities as

$$\chi(x_0, x_1, x_2, z, t_0, t_1, t_2, \varepsilon) = \varepsilon^2 \chi^{(1)} + \varepsilon^3 \chi^{(2)} + \ldots. \tag{9}$$

The perturbation equations of various orders so obtained are given in the Appendix.

3. Linear and Nonlinear Analysis

If we neglect the second harmonic resonance, the linear theory yields the dispersion relation

$$D(\omega, k) = (1 - \varrho) + k(1 + \varrho H^2) - \frac{\omega^2}{k} - \frac{\varrho}{k}(\omega - kU)^2 + \sigma k^2, \tag{10}$$

where ω and k denote the frequency and wavenumber of the fundamental mode only and $\varrho = \varrho^{(2)}/\varrho^{(1)}$, $H = H_2/H_1$. We dimensionalize the various physical quantities with respect to a characteristic length, characteristic time and characteristic surface tension defined by $h^{(1)^2}/g$, $h^{(1)}/g$, and $\varrho^{(1)} h^{(1)^4}/g$. To describe the second harmonic resonance on the marginally neutral curve obtained by (10), it is required that, in addition to the fundamental mode $A_1(x_1, x_2, t_1, t_2) \cdot \exp[i(k x_0 - \omega t_0)]$, we must take into account the harmonic mode $A_2(x_1, x_2, t_1, t_2) \exp[i(k_2 x_0 - \omega_2 t_0)]$, where $\omega_2 = 2\omega_1 + o(\varepsilon)$ and $k_2 = 2k_1 + o(\varepsilon)$.

The second order problem, governed by (A 7)–(A 13), admits the solutions

$$\eta_1 = \sum_{n=1}^{2} A_n e^{\imath \theta_n} + \text{c.c.}, \tag{11}$$

$$(u_1^{(j)}, w_1^{(j)}) = (\lambda_j, -\imath) \sum_{n=1}^{2} \Omega_{nj} A_n e^{\delta_{nj}} + \text{c.c.}, \tag{12}$$

$$(h_{x1}^{(j)}, h_{z1}^{(j)}) = (+\lambda_j H_j, \imath H_j) \sum_{n=1}^{2} k_{nj} A_n e^{\delta_{nj}} + \text{c.c.}, \tag{13}$$

$$\Pi_1^{(j)} = \frac{\lambda_j}{k} \sum_{n=1}^{2} [\Omega_{nj}^2 - k_n^2 H_j^2] A_n e^{\delta_{nj}} + \text{c.c.}, \tag{14}$$

where $\Omega_{nj} = (\omega_n - k_n U^{(j)})$; $\delta_{nj} = \imath\theta_n + \lambda_j k_n z$; $\theta_n = k_n x_0 - \omega_n t_0$. Also, for $j = 1, 2$ we define $\lambda_1 = 1$, $\lambda_2 = -1$; $U^{(1)} = 0$, $U^{(2)} = U$; $H_1 = 1$ and $H_2 = H$. Now the dispersion relation is of the form

$$\omega_n^2 + \varrho(\omega_n - k_n U)^2 - k_n^2(1 + \varrho H^2) - (1 - \varrho) k_n - \sigma k_n^3 = 0. \tag{15}$$

The resonant harmonic wavenumber can be derived from this dispersion relation by letting $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$. The value of the fluid velocity U at the marginally neutral curve is given by

$$U_m = \left[\frac{(1+\varrho)}{\varrho^{(2)} k_m}(\varrho^{(1)} k_m (H_1^2 + \varrho H_2^2) + \varrho^{(1)} g(1-\varrho) + \sigma k_m^2)\right]^{1/2}, \tag{16}$$

where the marginally stable wavenumber k_m is defined by

$$k_m = \left[\frac{g(\varrho^{(1)} - \varrho^{(2)})}{2\sigma}\right]^{1/2}. \tag{17}$$

The system is unstable for $U > U_m$. It is clear from (16) that U_m depends on the magnetic fields H_1 and H_2. It may be of interest to examine the characteristic of the second harmonic resonance compared with the other nonlinear processes from the choice of the scaling used in this paper. It is expected that the nonlinear coupling due to the second harmonic resonance shall play an efficient role to transfer energy from the mean flow to the system.

We now substitute the solutions from (11)–(17) into the third order problem governed by (A 14)–(A 20). After some straight forward algebra we obtain the following non-secularity conditions on the marginally neutral curve:

$$\frac{\partial A_n}{\partial x_1} = 0, \quad n = 1, 2. \tag{18}$$

This equation implies that the amplitudes A_n are independent of the faster scale x_1, but may depend upon x_2, t_1 and t_2.

We now substitute the second and third order equations into the fourth order equations (A 21)–(A 27). Taking into consideration only terms upto $\exp(\imath\theta_n)$ and ignoring others, which are of no interest, we obtain

$$L_{10}(u_3^{(j)}, \Pi_3^{(j)}, h_{x3}^{(j)}) = \sum_{n=1}^{2}\left[\imath w_{nj} - \lambda_j\left(S_{nj} + \frac{P_{nj}}{k_n} N_2(A_n)\right)\right] e^{\delta_{nj}} + 2\imath(k_1 - k_2) R_j A_2 \bar{A}_1 e^{\imath\phi_1} + \text{c.c.}, \tag{19}$$

$$L_{20}(w_3^{(j)}, \Pi_3^{(j)}, h_{z3}^{(j)}) = \sum_{n=1}^{2} [\imath S_{nj} - Y_{nj}] e^{\delta_{nj}} - 2\lambda_j (k_1 + k_2) R_j A_2 \bar{A}_1 e^{\imath\phi_1} + \text{c.c.}, \tag{20}$$

$$L_{30}(h_{x3}^{(j)}, u_3^{(j)}) = H_j \sum_{n=1}^{2} [\lambda_j (V_{nj} - \imath M_{1j}(M_{1j}(A_n))) \pm \imath k_n^2 U_m^j A_n] e^{\delta_{nj}} + 2\imath(k_1 + k_2) T_j H_j A_2 \bar{A}_1 e^{\imath\phi_1} + \text{c.c.}, \tag{21}$$

$$L_{40}(h_{z3}^{(j)}, w_3^{(j)}) = H_j \sum_{n=1}^{2} [\imath H_j (V_{nj} \pm \imath k_n^2 U_m^j A_n] e^{\delta_{nj}} - 2\imath(k_1 - k_2) T_j \lambda_j A_2 \bar{A}_1 e^{\imath\phi_1} + \text{c.c.}, \tag{22}$$

$$L_{50}(u_3^{(j)}, w_3^{(J)}) = -\lambda_j \sum_{n=1}^{2} w_{nj} N_2(A_n) e^{\delta_{nj}} + \text{c.c.}, \tag{23}$$

$$L_{60}(h_{x3}^{(j)}, h_{z3}^{(j)}) = \lambda_j H_j \sum_{n=1}^{2} k_n N_2(A_n) e^{\delta_{nj}} + \text{c.c.}, \tag{24}$$

$$L_{70}(\eta_3, w_3^{(j)}) = -\sum_{n=1}^{2} [M_{2j}(A_n) \pm \imath U_m^{(j)} k_n A_n] e^{\delta_{nj}} + 2\imath k_1 \Omega_{1j} A_1^2 e^{\imath\theta_2} - \imath(k_1 + k_2)(\Omega_{1j} - \Omega_{2j}) A_2 \bar{A}_1 e^{\imath\theta_1} + \text{c.c.}, \tag{25}$$

$$L_{80}(\eta_3, h_{z3}^{(j)}) = -H_j \sum_{n=1}^{2} N_2(A_n) e^{\delta_{nj}} - 2\imath k_1^2 H_j A_1^2 e^{\imath\theta_2} + \text{c.c.}, \tag{26}$$

$$L_9(\Pi_3^{(1)}, \Pi_3^{(2)}, \eta_3) = -2\imath\sigma \sum_{n=1}^{2} k_n N_2(A_n) e^{\imath\theta_n} + \sum_{n=1}^{2} (-1)^n \varrho^n [(P_{1n} + P_{2n}) A_2 \bar{A}_1 e^{\imath\theta_1} + P_{1n} A_1^2 e^{\imath\theta_2}] + \text{c.c.}, \tag{27}$$

where

$$P_{nj} = \Omega_{nj}^2 - k_n^2 H_j^2; \quad \phi_1 = \theta_1 - \imath\lambda_j (k_1 + k_2) z;$$
$$R_j = \Omega_{1j}\Omega_{2j} - k_1 k_2 H_j^2; \quad T_j = \Omega_{1j} k_2 - \Omega_{2j} k_1;$$
$$S_{nj} = \Omega_{nj} M_{2j}(A_n) + k_n H_j^2 N_2(A_n);$$
$$V_{nj} = k_n M_{2j}(A_n) + \Omega_{nj} N_2(A_n);$$
$$W_{nj} = -M_{1j}(M_{1j}(A_n)) \pm U_m^{(j)} k_n \Omega_{nj} A_n;$$
$$Y_{nj} = M_{1j}(M_{1j}(A_n)) \pm U_m^{(j)} k_n \Omega_{nj} A_n. \tag{28}$$

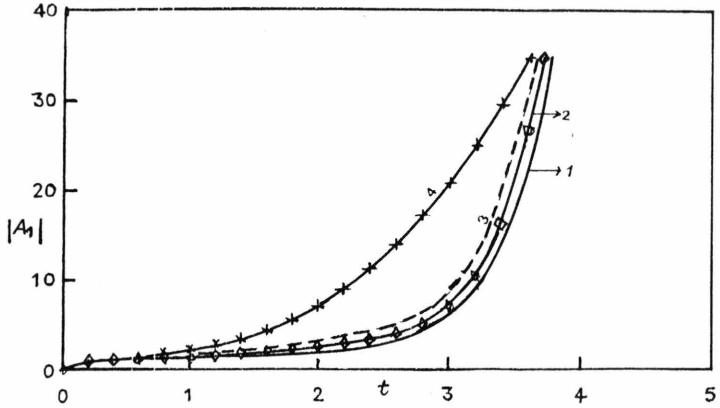

Fig. 1. The behaviour of $|A_1|$ as a function of time for the density ratio $\varrho = 0.4929$ and varying H^2. The curves 1, 2, 3, and 4 refer to $H^2 = 0$, 1, 2, and 5 respectively.

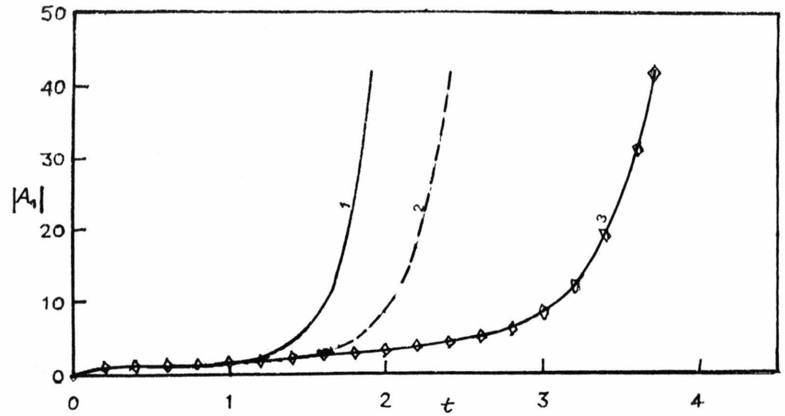

Fig. 3. The behaviour of $|A_1|$ as a function of time for $H^2 = 2$ and varying density ratio ϱ. The curves 1, 2, and 3 in the figure refer to $\varrho = 0.1$, 0.25 and 0.4929, respectively.

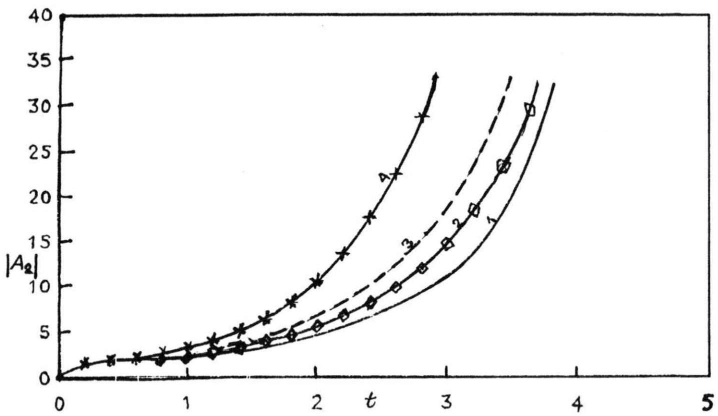

Fig. 2. The behaviour of $|A_2|$ as a function of time for the density ratio $\varrho = 0.4929$ and varying H^2. The curves 1, 2, 3, and 4 refer to $H^2 = 0$, 1, 2, and 5 respectively.

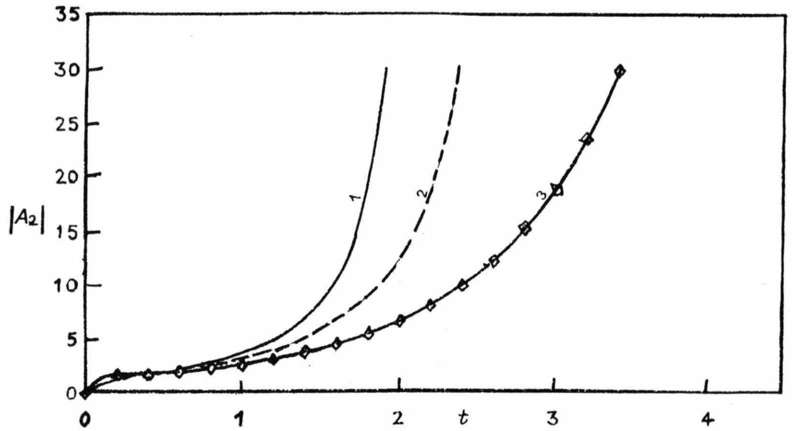

Fig. 4. The behaviour of $|A_2|$ as a function of time for $H^2 = 2$ and varying density ratio ϱ. The curves 1, 2, and 3 in the figure refer to $\varrho = 0.1$, 0.25 and 0.4929, respectively.

After some straight forward reductions it is found that the fourth order solutions are uniformly valid subject to the following conditions:

$$\frac{1}{2} D_{\omega_1 \omega_1} \frac{\partial^2 A_1}{\partial t_1^2} + \imath D_{k_1} \frac{\partial A_1}{\partial x_2} = \pm D_u A_1 + N_1 A_2 \bar{A}_1, \quad (29)$$

$$\frac{1}{2} D_{\omega_2 \omega_2} \frac{\partial^2 A_2}{\partial t_1^2} + \imath D_{k_2} \frac{\partial A_2}{\partial x_2} = \pm D_u A_2 + N_1 A_1^2, \quad (30)$$

where the $\pm$ signs of the linear terms coincide with $U = U_m \pm \varepsilon^2$.

The coefficients N_1 and N_2 are given by

$$N_1 = 2[(\omega_1^2 - k_m^2 H_1^2) - \varrho[(\omega_1 - k_1 U_m)^2 - k_m^2 H_2^2]], \quad (31)$$

$$N_2 = N_1/2. \quad (32)$$

It appears that it is not easy to solve (29) and (30) quite generally. We find that these equations reduce to ordinary differential equations if we assume that the spatial dependence of the amplitude A is negligible. We now apply numerical integration techniques to investigate the nature of the solutions. We consider fluids with the densities $\varrho^{(2)} = 0.7813$ and $\varrho^{(1)} = 1.585$ (see Chandrasekhar [6]) with the initial conditions

$$\mathrm{Re}(A_1) = 1; \quad \mathrm{Im}(A_1) = 0.5;$$
$$\mathrm{Re}(A_2) = -0.5; \quad \mathrm{Im}(A_1) = -1.5. \quad (33)$$

The initial speeds of A_1 and A_2 are taken to be zero. Typical solutions are shown in Figures 1–4. In Figs. 1–2, we have plotted the graph of the time development of the amplitudes, for fixed density ratio of the two fluids mentioned above and varying H^2, while in Figs. 3–4, we have plotted similar diagrams for varying density ratios and $H^2 = 2.0$. These diagrams show that the amplitudes $|A_1|, |A_2|$ grow unboundedly in finite time, thus leading to explosive instability near the second harmonic resonance.

4. Conclusions

The solutions of coupled amplitude equations reveal that at a certain time the amplitudes of both the fundamental mode and the second harmonic mode start increasing sharply, resulting in what is termed "Explosive Instability". It is demonstrated that for fluids of given density ratio, ϱ, the magnetic field plays an important role. The explosive instability sets in much earlier as the magnetic field increases. On the other hand, it is observed that for a given magnetic field, the explosive instability takes place at a slower rate for fluids of higher density ratio.

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Appendix

Let us define the following operators:

$$L_k[u_j^{(i)}] = \left[\frac{\partial}{\partial t_k} + U^{(i)} \frac{\partial}{\partial x_k}\right] u_j^{(i)}, \quad (\mathrm{A}\,1)$$

$$M_k(p) = \frac{\partial p}{\partial x_k}; \quad D(p) = \frac{\partial p}{\partial z}, \quad (\mathrm{A}\,2)$$

$$S_k^{(i)}[\eta_j] = \left[\frac{\partial}{\partial t_k} + U^{(i)} \frac{\partial}{\partial x_k}\right] \eta_j, \quad (\mathrm{A}\,3)$$

$$R_k^{(i)} = \left[\frac{\partial}{\partial t_k} + U^{(i)} \frac{\partial}{\partial x_k}\right], \quad (\mathrm{A}\,4)$$

$$\nabla_i \eta_k = \frac{\partial^2 \eta_k}{\partial x_0 \partial x_i}; \quad \nabla_i^2 \eta_k = \frac{\partial^2 \eta_k}{\partial x_i^2}, \quad (\mathrm{A}\,5)$$

$$Q[n] = \pm U_m \frac{\partial n}{\partial x_0}. \quad (\mathrm{A}\,6)$$

The equations of the order $o(\varepsilon^2)$ are

$$L_0[u_1^{(i)}, w_1^{(i)}] + [M_0, D]\Pi_1^{(i)} - H_i M_0[h_{x1}^{(i)}, h_{z1}^{(i)}] = 0, \quad (\mathrm{A}\,7)$$

$$L_0[h_{x1}^{(i)}, h_{z1}^{(i)}] - H_i M_0[u_1^{(i)}, w_1^{(i)}] = 0, \quad (\mathrm{A}\,8)$$

$$M_0[u_1^{(i)}] + D[w_1^{(i)}] = 0, \quad (\mathrm{A}\,9)$$

$$M_0[h_{x1}^{(i)}] + D[h_{z1}^{(i)}] = 0, \quad (\mathrm{A}\,10)$$

with the boundary conditions

$$w_1^{(i)} - S_0^{(i)}[\eta_1] = 0, \quad (\mathrm{A}\,11)$$

$$h_{z1}^{(1)} - H_1 M_0[\eta_1] = h_{z1}^{(2)} - H_2 M_0[\eta_1], \quad (\mathrm{A}\,12)$$

$$\varrho^{(1)}[\Pi_1^{(1)} - g\eta_1] - \varrho^{(2)}[\Pi_1^{(2)} - g\eta_1] + \sigma \nabla_0^2 \eta_1 = 0. \quad (\mathrm{A}\,13)$$

The equations of order $o(\varepsilon^3)$ are

$$L_0[u_2^{(i)}, w_2^{(i)}] + [M_0, D]\Pi_2^{(i)} - H_i M_0[h_{x2}^{(i)}, h_{z2}^{(i)}] = -L_1[u_1^{(i)}, w_1^{(i)}] - [M_1, 0]\Pi_1^{(i)} + H_i M_0[h_{x1}^{(i)}, h_{z1}^{(i)}], \quad (A14)$$

$$L_0[h_{x2}^{(i)}, h_{z2}^{(i)}] - H_i M_0[u_2^{(i)}, w_2^{(i)}] = -L_1[h_{x1}^{(i)}, h_{z1}^{(i)}] + H_i M_1[u_1^{(i)}, w_1^{(i)}], \quad (A15)$$

$$M_0[u_2^{(i)}] + D[w_2^{(i)}] = -M_1[u_1^{(i)}], \quad (A16)$$

$$M_0[h_{x2}^{(i)}] + D[h_{z2}^{(i)}] = -M_1[h_{x1}^{(i)}], \quad (A17)$$

with the boundary conditions

$$w_2^{(i)} - S_0^{(i)}[\eta_2] = S_1^{(i)}[\eta_1], \quad (A18)$$

$$h_{z2}^{(1)} - H_1 M_0[\eta_2] - H_1 M_1[\eta_1] = h_{z2}^{(2)} - H_2 M_0[\eta_2] - H_2 M_1[\eta_1], \quad (A19)$$

$$\varrho^{(1)}[\Pi_2^{(1)} - g\eta_2] - \varrho^{(2)}[\Pi_2^{(2)} - g\eta_2] + \sigma V_0^2\eta_1 = -2\sigma V_1\eta_1. \quad (A20)$$

The equations of order $o(\varepsilon^4)$ are

$$\begin{aligned} L_0[u_3^{(i)}, w_3^{(i)}] &+ [M_0, D]\Pi_3^{(i)} - H_i M_0[h_{x3}^{(i)}, h_{z3}^{(i)}] = -L_1[u_2^{(i)}, w_2^{(i)}] - L_2[u_1^{(i)}, w_1^{(i)}] - [M_1, 0]\Pi_2^{(i)} \\ &- [M_2, 0]\Pi_1^{(i)} + H_i M_1[h_{x2}^{(i)}, h_{z2}^{(i)}] + H_i M_2[h_{x1}^{(i)}, h_{z1}^{(i)}] - u_1^{(i)} M_0[u_1^{(i)}, w_1^{(i)}] - w_1^{(i)} D[u_1^{(i)}, w_1^{(i)}] \\ &+ h_{x1}^{(i)} M_0[h_{x1}^{(i)}, h_{z1}^{(i)}] + h_{z1}^{(i)} D[h_{x1}^{(i)}, h_{z1}^{(i)}] - Q[u_1^{(i)}], \end{aligned} \quad (A21)$$

$$\begin{aligned} L_0[h_{x3}^{(i)}, h_{z3}^{(i)}] &- H_i M_0[u_3^{(i)}, w_3^{(i)}] = -L_1[h_{x2}^{(i)}, h_{z2}^{(i)}] - L_2[h_{x1}^{(i)}, h_{z1}^{(i)}] + H_i M_1[u_2^{(i)}, w_2^{(i)}] + H_i M_2[u_1^{(i)}, w_1^{(i)}] \\ &- u_1^{(i)} M_0[h_{x1}^{(i)}, h_{z1}^{(i)}] - w_1^{(i)} D[h_{x1}^{(i)}, h_{z1}^{(i)}] + h_{x1}^{(i)} M_0[u_1^{(i)}, w_1^{(i)}] + h_{z1}^{(i)} D[u_1^{(i)}, w_1^{(i)}] - Q[h_{x1}^{(i)}], \end{aligned} \quad (A22)$$

$$M_0[u_3^{(i)}] + D[w_3^{(i)}] = -M_1[u_2^{(i)}] - M_2[u_1^{(i)}], \quad (A23)$$

$$M_0[h_{x3}^{(i)}] + D[h_{z3}^{(i)}] = -M_1[h_{x2}^{(i)}] - M_2[h_{x1}^{(i)}], \quad (A24)$$

with the boundary conditions

$$w_3^{(i)} - S_0^{(i)}[\eta_3] = S_1^{(i)}[\eta_2] + S_2^{(i)}[\eta_1] + Q[\eta_1] + u_1^{(i)} M_0[\eta_1] - \eta_1 D[w_1^{(i)}], \quad (A25)$$

$$\begin{aligned} h_{z3}^{(1)} &- H_1 M_0[\eta_3] - H_1 M_2[\eta_1] - H_1 M_1[\eta_2] + h_{x1}^{(1)} M_0[\eta_1] - \eta_1 D[h_{z1}^{(1)}] \\ &= h_{z3}^{(2)} - H_2 M_0[\eta_3] - H_2 M_2[\eta_1] - H_2 M_1[\eta_2] + h_{x1}^{(2)} M_0[\eta_1] - \eta_1 D[h_{z1}^{(2)}], \end{aligned} \quad (A26)$$

$$\begin{aligned} \varrho^{(1)}&[\Pi_3^{(1)} - g\eta_3] - \varrho^{(2)}[\Pi_3^{(2)} - g\eta_3] + \sigma V_0^3\eta_3 \\ &= -\sigma[2V_1\eta_2 + 2V_2\eta_1 + V_1^2\eta_1] - \eta_1[\varrho^{(1)} D[\Pi_1^{(1)}] - \varrho^{(2)} D[\Pi_1^{(2)}]]. \end{aligned} \quad (A27)$$

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